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Entrance examination – Question sheet (Mathematics)
Space and Astronautical Science Program, Graduate Institute for Advanced Studies, SOKENDAI

Question 1 Answer the following questions regarding the following matrices *A, B,* and *C*. Write the answer as well as the process of derivation.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}$$

Question 1-1. Determine the respective ranks of the matrices *A, B,* and *C*. If the matrices are regular, find the respective inverse matrices.

Question 1-2. Find the respective eigenvalues and the characteristic vectors of the matrices *A, B,* and *C*. The characteristic vectors must be normalized.

Question 1-3. A vector $\boldsymbol{r}$ on a plane can be expressed using two linearly independent vectors $\boldsymbol{e_1}$ *and* $\boldsymbol{e_2}$ on the plane and constants c_1 *and* c_2 as follows:

$$\boldsymbol{r} = c_1\boldsymbol{e_1} + c_2\boldsymbol{e_2}$$

Illustrate $\boldsymbol{r}$ and $A\boldsymbol{r}$ when the linearly independent vectors $\boldsymbol{e_1}$ *and* $\boldsymbol{e_2}$ are the normalized characteristic vectors of the matrix *A*, which are found in Question 1-2.

Question 1-4. Illustrate the figure expressed by the following quadratic equation in the x-y coordinate system:

$$3x^2 + 2xy + 3y^2 = 4$$

considering the followings (where ${}^tA$ is the transpose of the matrix *A*)

- If $F(x,y) = 3x^2 + 2xy + 3y^2$ and $\boldsymbol{x} = {}^t(x \quad y)$, $F = {}^t\boldsymbol{x}B\boldsymbol{x}$ holds. (The matrix *B* is the matrix defined at the beginning of Question 1.)
- Use coordinate transformation to make $\boldsymbol{y} = {}^t(X \quad Y) = P^{-1}\boldsymbol{x}$, where *P* is the orthogonal matrix to diagonalize the matrix *B*.

(This material is collected after the exam.)

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Question 2

Question 2-1. Find the derivative of the following function, where x is a real number. Write the answer as well as the process of derivation for questions (3), (4) and (5).

(1) $f(x) = (x^2 + 1)^3$

(2) $f(x) = \sqrt{x}$

(3) $f(x) = \tan x$

(4) $f(x) = \cos^{-1} x \ (-1 < x < 1, 0 < \cos^{-1} x < \pi)$

(5) $f(x) = x^x$

Question 2-2. Find the following indefinite integral $F(x)$, where x is a real number. Write the answer as well as the process of derivation for questions (2) and (3).

(1) $F(x) = \int (x-3)^3 dx$

(2) $F(x) = \int \cos^2 x dx$

(3) $F(x) = \int \frac{1}{\sqrt{x^2+2}} dx$

Question 2-3. Prove the following inequality, where x is a real number.

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{36} \qquad (x \neq 0)$$

Question 2-4. Find the general solution of the following differential equation, where $x, y, and\ p$ are real numbers. Write the answer as well as the process of derivation.

(1) $x(x-y)\frac{dy}{dx} + y^2 = 0$

(2) $p^2 + 2xp - 3x^2 = 0, \ p = y'$

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Question 2-5. Prove the following transformation of operators (1) and (2), where x is a real number, $f(x)$ is a continuously differentiable function, and n is a natural number.

(1) $e^{f(x)} \frac{d}{dx} e^{-f(x)} = \frac{d}{dx} - f'(x)$

(2) $e^{f(x)} \frac{d^n}{dx^n} e^{-f(x)} = \left(\frac{d}{dx} - f'(x)\right)^n$

The following function of a variable x:

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$$

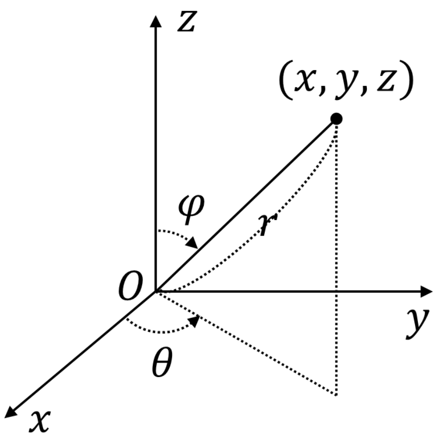
is known as the Hermite polynomial. Regarding this, prove (3), (4), and (5).

(3) $(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x)$

(4) $H_n'(x) = H_{n-1}(x)$

(5) $H_n''(x) - xH_n'(x) + nH_n(x) = 0$

Question 2-6. In a three-dimensional space, consider the transformation from the orthogonal coordinate system (x, y, z) to the polar coordinate system (r, θ, φ) shown in the right figure.



(1) Find the following Jacobian matrix J of this coordinate transformation.

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)}$$

(2) Find the following integral using the above Jacobian matrix, where $a > 0$.

$$\iiint_D x^2 dxdydz \qquad D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqq a^2\}$$

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