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Entrance examination – Question sheet (Mathematics)
Department of Space and Astronautical Science,
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Question 1-1 Consider transformation including liner transformation and translation of a point on a two-dimensional plane (x-y plane $[E^2]$ here), i.e.,

$$\mathrm{X} \mid \rightarrow \mathrm{A}\boldsymbol{x} + \mathbf{m}$$

where A is a linear transformation matrix and $\boldsymbol{x}$ *and* $\mathbf{m}$ are column vectors expressing coordinates and the amount of translation of the point, respectively.
For example, a point $a(a_1, a_2)$ is transformed to $b(b_1, b_2)$ by a linear transformation matrix, $\mathrm{A} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and translation, $\mathrm{m}(m_1, m_2)$, it can be expressed using matrices as follows:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

Also, it can be expressed as follows:

$$\begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix} = \begin{pmatrix} p & q & m_1 \\ r & s & m_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ 1 \end{pmatrix}$$

The values in the third row of the column vectors are meaningless for coordinate transformation, however, the transformation matrix becomes a square matrix, which enables to express linear transformation and translation as a single matrix and handle it in the same manner as linear transformation. Let the above 3x3 square matrix be A_f.
Show the matrix A_f expressing the following transformation described in (1) to (3).
Also, show that the matrix A_f is invertible according to the instructions in the following questions and find the inverse matrix A_f^{-1}.

(1) Show the transformation matrix A_f expressing movement of a point $a(a_1, a_2)$ by +2 in the x-direction and -3 in the y-direction. Show that A_f is invertible when $Ker\ \mathrm{A}_f = \{0\}$. Find the inverse matrix A_f^{-1}.

(2) Show the transformation matrix A_f expressing counterclockwise rotation of a

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point $a(a_1, a_2)$ around the origin (0,0) by an angle $\theta(rad)$. Show that A_f is invertible when $\det|\mathrm{A}_f| \neq 0$. Find the inverse matrix A_f^{-1}.

(3) Show the transformation matrix A_f expressing counterclockwise rotation of a point $a(a_1, a_2)$ around the origin (0,0) by an angle $\theta(rad)$ followed by movement by +2 in the x-direction and -3 in the y-direction. Show that A_f is invertible when rank=3. Find the inverse matrix A_f^{-1}.

Question 1-2 Consider transformation of a square $(\mathrm{E}(-1,-10), \mathrm{F}(-1,4), \mathrm{G}(7,-2), \mathrm{H}(7,-4))$
by rotating it around the center of gravity $\mathrm{G}(g_1, g_2)$ by $\pi/4\,(rad)$ clockwise.

（1） Find the coordinates of the center of gravity (g_1, g_2) of this square. "The center of gravity" is a point where the square balances when it consists of uniform material.

（2） Find the transformation matrix A_f. When you can find the coordinates of the center of gravity in Question (1), use the coordinates. If not, you can let the coordinate be $\mathrm{G}(g_1, g_2)$ for calculation.

Question 1-3 Define the function e^{A}, which is derived by replacing x (real number) of the exponential function e^x with a square matrix of order n, as follows:

$$e^{\mathrm{A}} = exp(\mathrm{A}) := \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{A}^k \ (\mathrm{A} \in \mathbf{R}^{\mathrm{n}})$$

where $A^0 = \mathrm{E}$ (identity matrix). It means that x is replaced with a matrix A in Maclaurin series expansion of the exponential function. Therefore, e^A is a square matrix of order n.

Maclaurin series expansion is Taylor series expansion at $x = 0$ shown as follows:

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!} = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots$$

For the exponential function e^x,

$$e^x = \sum_{k=0}^{\infty} e^{(k)}(0) \frac{x^k}{k!} = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots$$

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（1） Find $exp(\mathrm{R})$ letting R be as follows. Explain the meaning of the derived matrix.

$$\mathrm{R} = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \quad x \in \mathrm{R}$$

（2） Show that the following equation is valid when the matrix G is a square matrix of order n similarly to the matrix A.

$$exp(\mathrm{G}^{-1}\mathrm{AG}) = G^{-1}exp(\mathrm{A})\mathrm{G}$$

（3） Find $exp(\mathrm{A})$ when the matrix A is as follows:

$$\mathrm{A} = \begin{pmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$

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Question 2-1　Find the derivative $\frac{df}{dx}$ of the function $f(x)$.

（1）$f(x) = \frac{(1-4x^2)^2}{1+x^2}$

（2）$f(x) = x^3 log(x^2)$

（3）$f(x) = \frac{sin\,x}{1+tan\,x}$

（4）$f(x) = cos^{-1}(x\sqrt{1-x^2})$

Question 2-2　Find the indefinite integral $\int f(x)\,dx$ of the function $f(x)$.

（1）$f(x) = (x-1)^4$

（2）$f(x) = sin^2 x$

（3）$f(x) = x^2 e^x$

Question 2-3　Solve the following differential equations.

（1）$\frac{dy}{dx}$=$3x\,(y-3)^2$

（2）$\frac{dy}{dx}$=2y+1

（3）$\frac{d^2y}{dx^2} - 10\frac{dy}{dx}$+74y=0

Question 2-4　Answer the following questions about the following function:

$$B(\nu, T) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

where c, h, and k are constants.

（1）Find $\frac{\partial B(\nu,T)}{\partial T}$.

（2）Find $\int_0^\infty B(\nu, T)d\nu$, where

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}$$

（3）Show $\int_0^\infty \frac{x^3}{e^x-1}\,dx = \frac{\pi^4}{15}$, where

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$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

（Hint）

Calculate the following function:

$$\sum_{n=1}^{\infty} e^{-nx}$$

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