Plasma Dispersion for non-Maxwellian Type Velocity Distribution: Waves and Sheath

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Effects of non-Maxwellian velocity distribution, e.g. Druyvesteyn-type, on the plasma dispersion are investigated and plasma waves and the sheath formation criterion are considered. Dispersions of ion acoustic and electron waves, the floating potential and Debye length are calculated and the differences from the case of Maxwellian distribution are described. Results may be useful for plasma diagnostics using probes and waves.

1. Introduction

In weakly ionized plasmas, the velocity distribution of electrons often deviates from Maxwellian to Druyvesteyn type [1]. The solution of Boltzmann equation showed that if the mean free path λ is constant, Druyvesteyn type is obtained [2]. Later, by expressing the relation between the elastic cross section σ and the energy E by $\sigma \propto E^n$ it was shown that if n=−1/2 (mean collision time τ is constant) Maxwellian is obtained [3]. This does not hold for most gases except Hg. Results of measurements have shown that the velocity distribution is more like Druyvesteyn [4-6].

In the present paper, the plasma dispersion for Druyvesteyn-type is derived, from which wave dispersions and the sheath formation criterion are calculated and compared with those of Maxwellian.

2.Druyvesteyn-type velocity distribution

The Druyvesteyn energy distribution [1] is given in an analytical form as [7]

$$F_D(E)=\frac{4\pi}{m}\sqrt{\frac{2E}{m}}\frac{a}{\pi(\frac{2E_m}{m})^{3/2}}\cdot\exp(-\frac{bE^2}{E_m{}^2}),\quad(2.1)$$

where E_m is the average energy ($=m<v^2>/2$), $a=\Gamma(5/4)^{3/2}/\Gamma(3/4)^{5/2}$, $b=\Gamma(5/4)^2/\Gamma(3/4)^2$ or a=0.5191, b=0.5471. Using the relation of the velocity f(v) and energy distributions F(E) [8],

$$f(v)=\frac{1}{2}\sqrt{\frac{m}{2}}\int_{eV}^{\infty}\frac{F(E)}{\sqrt{E}}dE,\qquad(2.2)$$

we obtain the 1-dim velocity distribution f(v). For the case of Druyvesteyn, we have

$$f_D(v)=\frac{\sqrt{\pi}}{2}\frac{a}{b^{1/2}<v^2>^{1/2}}\mathrm{Erfc}(\frac{b^{1/2}v^2}{<v^2>}),\quad(2.3)$$

The relation (2.2) recovers the relation between energy and velocity distributions for the case of Maxwellian.

This Druyvesteyn-type velocity distribution reflects well the characteristics of $F_D(E)$ well: the low and high velocity regions are more deficient while the medium region more populated than in Maxwellian velocity distribution.

3. Effect on the wave dispersions

The dispersion relation for the electron-ion plasma is given by

$$1+\frac{\omega_e{}^2}{k}\int_{-\infty}^{\infty}\frac{\partial f_e/\partial v}{\omega-kv}dv-\frac{\omega_i{}^2}{(\omega-kv)^2}dv=0,\ (3.1)$$

where standards notations are used. Here, we assume exp[i(ωt - kx)] contrary to Fried-Conte [10]: The pole is assumed in the upper half of x-axis for the damping mode.

3.1 Electron waves

Dropping the positive ion term, putting $x=b^{1/4}v/<v^2>^{1/2}$ and $\zeta=(\omega/k)b^{1/4}/<v^2>^{1/2}$ and substituting $f_D(v)$ into f_e in (3.1), we have

$$1-\frac{\omega_e{}^2}{k^2}\frac{2a}{b^{1/4}<v^2>}\int_{-\infty}^{\infty}\frac{x}{\zeta-x}\cdot\exp(-x^4)dx=0,\ (3.2)$$

Define $Y_D(\zeta)$ as

$$Y_D\equiv\int_{-\infty}^{\infty}\frac{x\exp(-x^4)}{x-\zeta}dx.\qquad(3.3)$$

We make asymptotic and series expansions.

(i)ζ>>x: phase velocity >> thermal velocity

$$Y_D\cong-\int_{-\infty}^{\infty}\exp(-x^4)(\frac{x^2}{\zeta^2}+\frac{x^4}{\zeta^4}+\frac{x^6}{\zeta^6}....)dx$$
$$-i\pi\zeta\exp(-\zeta^4)$$

The imaginary part is assumed to be small.

Using the relation $v_m^2=<v^2>/(2b)^{1/2}$, we obtain the dispersion relation to the second order,

$$\omega^2 \cong \omega_e^2[1+\frac{\sqrt{2b}\,k^2 v_m^2}{\omega^2}+...]$$
$$+i\pi\frac{\omega\omega_e^2}{k^3}(\frac{1}{a(2b)^{3/4}v_m^3})\exp(-\frac{\omega^4}{2k^4 v_m^4})]. \quad (3.4)$$

For comparison, the case of Maxwellian is

$$\omega^2 \cong \omega_e^2(1+\frac{3k^2 v_m^2}{2\omega^2}+\frac{15k^4 v_m^4}{2\omega^4}+...)$$
$$+i2\sqrt{\pi}(\frac{\omega_e^2\omega^3}{k^3 v_m^3})\exp(-\frac{\omega^2}{k^2 v_m^2}). \quad (3.5)$$

Figure 1 shows the dispersion relation ω/ω_e vs kv_m/ω_e for Druyvesteyn D and Maxwellian M distributions. It is seen that the group or phase velocity is smaller and the Landau damping rate is smaller for Druyvesteyn distribution.

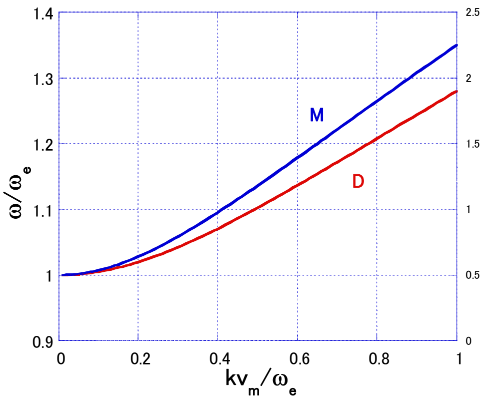

Fig.1. Dispersion of electron waves.

3.2 Ion acoustic waves

As the phase velocity is much smaller than v_m, we make series expansion of Y_D for $\zeta << x$

$$Y_D = \{2\Gamma(\frac{5}{4})-2\Gamma(\frac{3}{4})\zeta^2-\frac{2}{3}\Gamma(\frac{1}{4})\zeta^4+\sum_{n=3}^{\infty}\zeta^{2n}I_{2n}\}$$
$$-i\pi\zeta\exp(-\zeta^4)$$

Neglecting kv in the ion term in (3.1), we obtain the dispersion relation as

$$\frac{\omega^2}{k^2} \cong \frac{\omega_p^2\sqrt{2}\cdot b^{3/4}v_m^2}{4a\omega_e^2\Gamma(5/4)}[1+i\pi\zeta/2\Gamma(5/4)], \quad (3.5)$$

The ion acoustic velocity is obtained as $C_D=0.6914(m/M)^{1/2}v_m$, which is slower than that of Maxwellian, $C_M=0.707(m/M)^{1/2}v_m$ by about 2.2%. The damping rate in the long wavelength limit, $\omega_i/\omega= 0.504(m/M)^{1/2}$, is a smaller than that of Maxwellian [11] given by $(\pi m/8M)^{1/2}C_M= 0.627(m/M)^{1/2}$ by ~20%. This is considered to be due to the fact that $f_D(v)$ is more rounded at small v than $f_M(v)$. Figure 2 shows the dispersion of ion waves for Druyvesteyn and Maxwellian denoted by D and M, respectively.

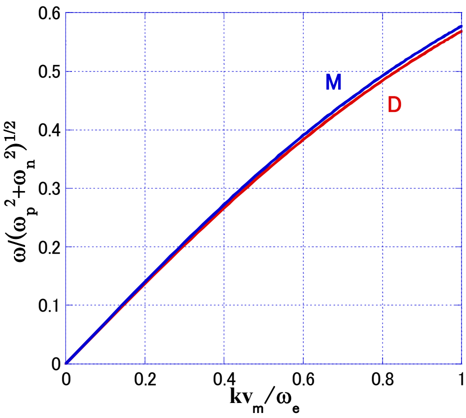

Fig. 2. Dispersion of ion waves.

4. Effect on probe characteristics

4.1 Electron current

The electron current density J_e in the retarding region of a plane probe is given by

$$J_e = \int_{\sqrt{-2eV_p/m}}^{\infty} vf(v)dv \quad (4.1)$$

Substituting f(v) from (2.2), we obtain for the Druyestyn distribution

$$J_e = J_o S_D[\exp(-b\eta^2)-\sqrt{\pi b}\eta\cdot\mathrm{erfc}(\sqrt{b}\eta)], \quad (4.2)$$

where $s_D=a/b=0.9488$ and J_o is defined as

$$J_o = \frac{n_o e<v^2>}{4} \quad (4.3)$$

For large η, the asymptotic form becomes

$$J_{e\,appx} = J_o S_D\exp(-b\eta^2)/(2b\eta^2) \quad (4.4)$$

which agrees with (4.2) with an error less than 5% above η>5.

For Maxwellian, we have the known form

$$J_e = J_o S_M\exp(-3\eta/2) \quad (4.5)$$

where $s_M=(8/2\pi)^{1/2}=0.9213$. J_oS_D and J_oS_D are the saturation current densities at η=0 for both velocity distributions, respectively.

Figure 3 shows the normalized electron current density $j_e=J_e/J_oS_D$ for Druyvesteyn with the asymptotic value j_{eappx} together with j_e=Je/$J_o S_M$ for Maxwellian distribution vs eV_p/E_m. It is seen that j_e of druyvesteyn decreases more steeply than j_e of Maxwellian, which is considered to be due to the lack of higher energy tail in $f_D(v)$.

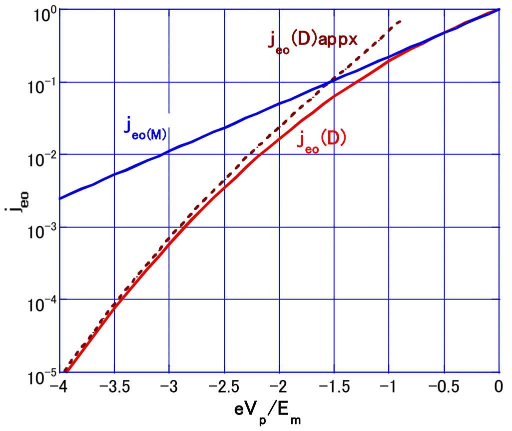

Fig.3. Electron current of plane probe in retarding region vs eV_p/E_m (=η). M:Maxwellian, D:Druyvesteyn.

3.2 Positive ion current

It is known that the sheath condition can be obtained from the dispersion relation [8],

$$1-\frac{\omega_e^{\ 2}}{k^2}\int_{-\infty}^{\infty}\frac{f_e}{v^2}dv-\frac{\omega_p^{\ 2}}{k^2}\int_{-\infty}^{\infty}\frac{f_p}{v_o^{\ 2}}dv=0 \qquad (4.6)$$

where f_e, f_p are the velocity distributions for electrons and positive ions respectively and ω_e, ω_p are their angular plasma frequencies.
In the limit of $k\lambda_D \to 0$ (the sheath thickness is much longer than Debye length), we have by neglecting k

$$\frac{1}{M}\int_{-\infty}^{\infty}\frac{\partial f_p/\partial v}{v}dv=\frac{1}{M}<v_o^{\ -2}>, \qquad (4.7)$$

where M is the ion mass. Using the relation between the one dimensional velocity distribution f(v) and the total energy distribution F(E) is [8],

$$\frac{\partial f(v)}{\partial v}=-\frac{m}{2}F(eV) \qquad (4.8)$$

we obtain

$$\frac{1}{M}<v_o^{\ -2}>=\int_0^{\infty}\frac{F_e(E)}{2E}dE. \qquad (4.9)$$

Substituting Druyvesteyn energy distribution, F_D (2.1), we have

$$\frac{\omega_p^{\ 2}}{<v_o^{\ 2}>}=\frac{2a\omega_e^{\ 2}}{b^{1/4}<v^2>}\int_{-\infty}^{\infty}\exp(-x^4)dx \qquad (4.10a)$$

i.e.

$$\frac{2<v_o^{\ -2}>}{M}=\frac{2\sqrt{2}\Gamma(5/4)}{\Gamma(3/4)}\frac{(2b)^{1/2}}{E_m}=\frac{2.19}{E_m} \qquad (4.10b)$$

For the case of Maxwellian (E_m= $3\kappa T_e/2$) [9],

$$\frac{2<v_o^{\ -2}>}{M}=\frac{2}{\kappa T_e}=\frac{3}{E_m} \qquad (4.11)$$

The reciprocal of LHS corresponds to the sheath edge potential V_s, which becomes $E_m/2.19$ and $E_m/3$ for Druyvesteyn and Maxwellian distributions respectively.
The positive ion current density J_+ at the sheath edge is equal to the ion acoustic velocity times the ion density there, i.e.

$$J_+=n_o e\delta_S\left(\frac{2eV_s}{M}\right)^{1/2} \qquad (4.12)$$

where $\delta_S=n_S/n_O$, n_S and n_O are the densities at the sheath edge and plasma, respectively. For a probe with a deep negative potential, all the electrons are reflected and δ_S is given by

$$\delta_S \equiv n_{es}/n_e=2\int_{v_S}^{\infty}\frac{vf(v)dv}{\sqrt{v^2-2eV_S/m}} \qquad (4.13)$$

where $v_s=(2eV_s/m)^{1/2}$. Integrating by part and using (4.8), (4.13) can be transformed into

$$\delta_S=\int_{eV_S}^{\infty}\sqrt{1-eV_s/E}\cdot F(E)dE \qquad (4.14)$$

Substituting V_s, δ_s is calculated to be 0.5648 for Druyvesteyn distribution, while δ_s= exp(-1/2)=0.6065 for Maxwellian. Then,

$$J_+=n_o e\,\mathrm{C}\left(\frac{2E_m}{M}\right)^{1/2} \qquad (4.15)$$

where C=0.3817 for Druyvesteyn and C=0.3502 for Maxwellian distributions respectively. J_+ is larger for Druyvesteyn distribution by about 9% for the same average energy E_m.

5. On the floating potential

5.1 DC floating potential

The DC floating potential is determined by the equating electron and positive ion current densities: $J_e=J_+$. Using (4.2) and (4.15) for J_e and J_+, we have

$$\frac{S_{D,M}}{4}j_e=C\sqrt{\frac{m}{M}} \qquad (5.1)$$

The normalized floating potential $\eta_f=eV_f/E_m$ was calculated numerically for Druyvesteyn distribution as a function of mass ratio M/m. In the case of Maxwellian, η_f states (ε=ln(1)),

$$\eta_f = \frac{1}{3}[\ln(M/m) + \ln(\varepsilon/2\pi)] \qquad (5.2)$$

Figure 4 shows η_f vs M/m for both velocity distributions. It is seen that is η_f is larger for Maxwellian by a factor of 1.34 (H_2) to 1.55 (Hg). This is considered to be due to the truncation of higher energy tail in $f_D(v)$, which gives smaller value of j_e at deeper negative biases.

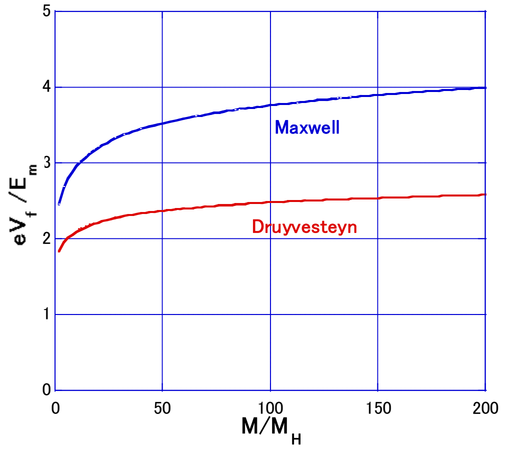

Fig. 4. Normalized floating potential vs M/m.

5.2 Effect of ac modulation

The substrate bias is modified by applying an alternating current voltage (ac) to it.

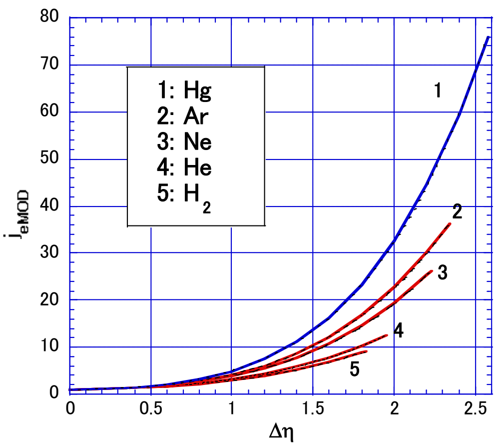

Fig. 5. Modulated current density j_{eMOD} vs the modulation amplitude $\Delta\eta$ for some gases.

The modulation effect depends on the velocity distribution given by $j_{eMOD}(\eta,\Delta\eta)$.

$$j_{eMOD} = \frac{1}{2\pi}\int_0^{2\pi} j_e(\eta + \Delta\eta \sin\omega t)d\omega t \qquad (5.3)$$

where $\Delta\eta$ is the amplitude of modulation normalized by E_m: $\Delta\eta=\Delta V/E_m$. This holds in the range $\eta>\Delta\eta$. If $\eta>\Delta\eta$; j_e in the phase of crossing $\eta=0$ is replaced by unity due to the assumption of an ideal planar probe.

For the case of Maxwellian, it is known that

$$j_{eMOD} = \exp(-3\eta/2)\mathrm{I_o}(3\Delta\eta/2) \qquad (5.4)$$

where I_o is the Bessel function of 0 th order.

Figure 5 shows the modulated current density j_{eMOD} as a function of $\Delta\eta$ for some gases in comparison with that of Maxwellian. It is seen that j_{eMOD} increases with $\Delta\eta$ and the increase rate is larger than for Maxwellian where j_{eMOD} is independent of the kind of gas, and that the modulation effect is slightly larger as the ion mass is larger.

6. Effect on Debye length

Substituting F(E) into the equation (4.14) for the density distribution at a potential V, we obtain the Poisson equation as

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) = \frac{n_o eC}{\varepsilon_o}\cdot\frac{2eV}{m{v_m}^2} \qquad (5.5)$$

where C=1 for F_M, C=1.046 for F_D. We get $\lambda_D=\varepsilon_o v_m^2/2Ce^2n_o$ for the potential

$$V \propto \frac{1}{r}\exp(-\frac{r}{\lambda_D}) \qquad (5.6)$$

It has been found that the Debye length for the Druyvesteyn distribution is 0.978 of that of Maxwellian distribution.

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