Entrance Examination
The Department of Space and Astronautical Science,
the School of Physical Sciences
the Graduate University for Advanced Studies
(Mathematics)

Question 1-1.

Consider the following indefinite integral:

$$I_n = \int (ln\ x)^n dx \qquad (1\text{–}1)$$

where n is an integer greater than or equal to 0, $ln\,x$ is the natural logarithm, and $x > 0$.

(1) Find the recurrence formula between I_{n+1} and I_n using integration by parts.

(2) Calculate I_3 using the result of (1). You do not need to determine the constant derived from the indefinite integral.

(3) Calculate the following definite integral, where e is the base of the natural logarithm.

$$\int_e^{e^2} (ln\ x)^3 dx \qquad (1\text{–}2)$$

Question 1-2.

For real numbers x, y, and z, let x, $y \geq 0$, and $z \geq 0$. Answer the following questions, including the reasons.

(1) When the sum of x and y is a constant value c, find the maximum value of the product $w = xy$ and (x, y) for the maximum value.

(2) When the sum of x, y, and z is a constant value c, find the maximum value of the product $w = xyz$ and (x, y, z) for the maximum value.

Question 2. Answer the following questions. Write formulas in $\boxed{(a)}$ to $\boxed{(l)}$ on the answer sheet. In the diagram, add representative points, etc. to give supplementary/additional explanations so that the intent of the solution can be understood. In the following, j represents an imaginary unit $\sqrt{-1}$.

(1) Let a certain point P in the coordinate system be given by (X, Y). When these are expressed by the components of the polar coordinate system, absolute value (or distance) $|P|$, and angle φ,

$$\mathrm{X} = |P|\cos\varphi$$
$$\mathrm{Y} = |P|\sin\varphi$$

Using Euler's formula, the complex representation is

$$\mathrm{P} = |P|(\cos\varphi + j\sin\varphi) = |P|e^{j\varphi} \qquad (2-1)$$

Here, consider the rotation operation of this point P. We are to find the coordinate value (X', Y') of point P' obtained by rotating point P around origin O in the positive direction by angle θ. When $\varphi+\theta$ is substituted for φ in equation (2-1), we obtain

$$\mathrm{P}' = |P|e^{j(\varphi+\theta)} = |P|\{\cos(\varphi+\theta) + j\sin(\varphi+\theta)\}$$

From the equation, using $\sin\varphi, \cos\varphi, \sin\theta, \cos\theta$ and $|P|$, we obtain

$$\mathrm{X}' = |P|\left\{\cos\varphi\cos\theta - \boxed{(a)}\right\}$$

$$\mathrm{Y}' = |P|\left\{\boxed{(b)} + \boxed{(c)}\right\}$$

(2) In the complex plane, define z_L and Γ as follows, where r, x, u, and v are real numbers, and θ is an angle parameter (unit: radian).

$$z_L = \mathrm{r} + j\mathrm{x} \qquad (2-2\mathrm{a})$$
$$\Gamma = \mathrm{u} + j\mathrm{v} = |\Gamma|e^{j\theta} \qquad (2-2\mathrm{b})$$

In the u-v coordinate system, equation (2-2b) draws a circle centered on the origin with a radius of $|\Gamma|$ (see Figure 2-0 in the answer sheet of Q2). Here, further translation in the u-v coordinate system is added, where z_L and Γ satisfy the following equation giving a conversion formula between the r - x coordinate system and the u - v coordinate system.

$$z_L = 1 + A\Gamma \qquad (2-3\mathrm{a})$$
$$A = |A|e^{j\varphi} \qquad (2-3\mathrm{b})$$

When equation (2-2) is substituted for equation (2-3) to express them as functions of u and v, we obtain

$$\mathrm{r} + j\mathrm{x} = \mathrm{r}(\mathrm{u},\mathrm{v}) + j\mathrm{x}(\mathrm{u},\mathrm{v}) = 1 + |A||\Gamma|e^{j(\varphi+\theta)} = \boxed{(d)} + j\boxed{(e)} \qquad (2-4)$$

These transformations map circles on the complex plane to circles using a composition of translations, similarities, and rotations (circle-to-circle transformation).

(3) Next, based on circle-to-circle transformation in the question above, consider linear fractional transformations for which inversions are added to the transformations above. Where the following z_L and Γ are defined by equation (2-2) in the question above, and z_L and Γ satisfy the following new equation giving a conversion formula between the r - x coordinate system and the u - v coordinate system.

$$z_L = \frac{1+\Gamma}{1-\Gamma} \qquad (2-5)$$

When equation (2-2) is substituted for equation (2-5) to express them as functions of u and v, we obtain

$$\mathrm{r} + j\mathrm{x} = \mathrm{r}(\mathrm{u},\mathrm{v}) + j\mathrm{x}(\mathrm{u},\mathrm{v}) = \boxed{(f)} + j\boxed{(g)} \qquad (2-6)$$

From the real part of this equation (2-6), in the u-v coordinate system using r as a parameter, the following formula of a circle is obtained.

$$\left(\boxed{(h)}\right)^2 + \mathrm{v}^2 = \left(\boxed{(i)}\right)^2 \qquad (2-7)$$

Write the coordinates of the circles when r = 0.5, 1.0, and 2.0, respectively. Also, show their diagrams in Figure 2-1 on the answer sheet. When r = 0, equation (2-7) represents a circle with a radius of 1 centered at the origin of the u-v coordinate system, and from equation (2-2b), $|\Gamma| = 1$ holds when u and v are arbitrary real numbers (i.e., the angle parameter θ).

In the same way as the derivation of equation (2-7), based on the imaginary part of equation (2-6), the equation of a circle in the u-v coordinate system using x as a parameter is obtained. In the same form as equation (2-7), we obtain

$$(\mathrm{u}-1)^2 + \left(\boxed{(k)}\right)^2 = \left(\boxed{(l)}\right)^2 \qquad (2-8)$$

Based on these, write the coordinates of the circles when x=±1.0 and ±2.0, respectively. Also, show their diagrams in the range of -1<u<1 and -1<v<1 in Figure 2-2 on the answer sheet, where x=0 corresponds to v=0 (u axis) in the u-v coordinate system.

(4) Figure 2-0, Figure 2-1, and Figure 2-2 are a group of curves of one graph which is different depending on each parameter in the u-v coordinate system. Based on this, simply explain the method to obtain x using Figure 2-0, Figure 2-1, and Figure 2-2, where $\mathrm{u} > 0, \mathrm{v} > 0$, $|\Gamma| = 0.5$, and r = 1.

The Graduate University for Advanced Studies, School of Physical Sciences, Space and Astronautical Science
Entrance Examination
(Physics)

Q1. A rigid body with a mass M is swinging as a rigid body pendulum around the fulcrum located at point O that is placed a distance d away from the gravity center G of the rigid body, as shown in Figure 1. Answer the following questions. Assume that the gravity acceleration is g and the inertial force of the rigid body around the fulcrum O is I.

(1) Let's consider the equation of motion that describes the rotation of the rigid body around the fulcrum O. Find an expression for the moment around the fulcrum O due to gravity when the angle between the line OG and the perpendicular direction is θ.

(2) Find an equation of motion that describes the rotation of the rigid body around the fulcrum O.

(3) Find the period T when the motion is a minute vibration by solving the equation of motion.

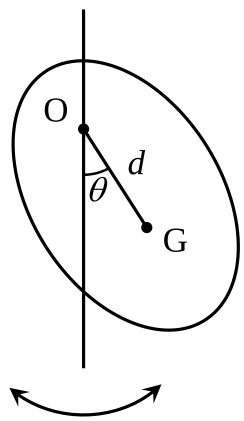

Figure 1

(4) $k$represents an amount that has a dimension of length. Describe I by using d, k, and M while assuming that the moment of inertia around the axis extended through the center of mass of the rigid body, G, is represented as Mk^2.

(5) When d is changed, the period T also changes. Draw a graph that represents the relation between d and T by assigning d and T to the horizontal and vertical axes respectively, and then find the minimum period T_{min} and d that gives the minimum period.

(6) There is a sufficiently thin bar and a sufficiently thin disk. They vibrate as a rigid body pendulum respectively. When the pendulums vibrate around the fulcrum that is respectively adjusted to the position that gives the minimum period, which pendulum vibrates in a shorter period? Assume that the length of the bar is equal to the diameter of the disk and that their masses are equal.

Q2. There is a small conductor globe (radius: a, mass: m) in a vacuum. Answer the following questions. For the following questions, use the units of the SI unit system and assume that the dielectric constant of the vacuum is ε_0 and the gravity acceleration is g.

Q2-1.

(1) Find the amount of energy (U) that is required to carry a charge $Q\ (> 0)$ from infinity to charge the globe. Assume that initially the conductor globe is not charged.

(2) There is no charge inside the charged conductor globe. Explain simply the reason for that.

(3) How the electrostatic energy changes when the circumference of the charged conductor globe is filled with a gas (dielectric constant: ε). Simply explain for what the change of the energy is used.

Q2-2.

The conductor globe on which the charge Q is applied is rested at a height h from the upper electrode of a set of parallel flat plate electrodes (a square shape with a side length L and a distance between the electrodes $d(\ll L)$) that is horizontally placed (Figure 2). The globe is free-fallen from the position to go through the space between the parallel flat plate electrodes from a small hole that is prepared on the upper surface of the electrodes. The electric potential difference V between the parallel flat plate electrodes can be arbitrarily set. Also a uniform magnetic field with a constant strength (magnetic flux density: B) can be applied to the space between the electrodes in the direction parallel to the electrode plates. The magnetic field can be turned on/off at any time. A sensor is installed in the hole on the upper electrode to detect the timing when the conductor globe goes through the hole.

Answer the following questions while assuming that the small hole on the upper electrode does not affect the free fall of the globe and the electric field between the parallel flat plate electrodes.

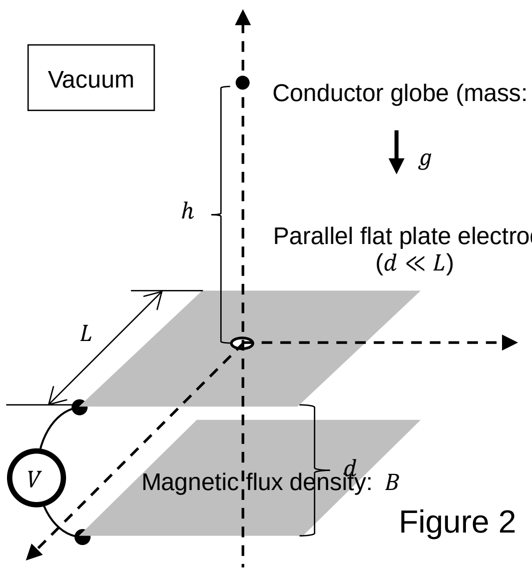

Figure 2

(1) Find the velocity v_0 of the conductor globe when it enters the hole on the upper surface of the parallel flat plate electrodes.

(2) Find the potential difference V that is required to be applied between the parallel flat plates to cancel the influence of gravity in the space between them.

(3) Consider a method of leading the conductor globe that has entered the space between the parallel flat plate electrodes out in the horizontal direction. Find the minimum value of the magnetic flux density B between the parallel flat plate electrodes that is required to prevent the conductor globe from contacting the lower electrode. Assume that the influence of gravity is canceled in the space between the parallel plates by the applied potential difference V.

(4) Describe how the magnetic field between the parallel flat plate electrodes should be turned on/off to lead the conductor globe out in the horizontal direction. Describe the process along with the time elapsed from when the conductor globe goes through the hole. Assume that the influence of gravity is canceled in the space between the parallel plates by the applied potential difference V.

(5) Assume that the mass of the conductor globe deviates by δm from the expected mass m or the charge deviates by δQ from the expected amount of charge Q. Describe how the track of the conductor globe changes with the mass that deviates by δm or with the charge that deviates by δQ. Assume that the conditions of the potential difference V, magnetic flux density B, and control for turning on/off the magnetic field that are set in above (4) for the conductor globe with the mass m and the charge Q do not change.